Note Title 11/3/2009

Electron in a Magnetic Field

Electron (or any particle with a spin) has a magnetic dipole moment

$$\vec{\mu} = \gamma \vec{S},$$

where γ is called the gyromagnetic ratio.

From electromagnetics, the energy of an electron placed in a magnetic field $\vec{B}$, is given by

$$H = -\vec{\mu}\cdot\vec{B}$$
$$= -\gamma \vec{B}\cdot\vec{S}$$

With $\vec{B} = B_0 \hat{k}$

$$\Rightarrow H = -\gamma B_0 S_z = -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

What are the energy eigenstates and eigenvalues?

$\boxed{\text{Ans:}}$ Eigenstates of H are the same as those of S_z

$$\Rightarrow \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow E_+ = -\gamma B_0 \left(\frac{\hbar}{2}\right)$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow E_- = -\gamma B_0 \left(-\frac{\hbar}{2}\right)$$

* If at $t=0$, the spin was in $\chi_+^{(x)}$ state what is its state at t ?

$$\chi(0) = \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\chi_+ + \chi_-)$$

$$\Rightarrow \chi(t) = \frac{1}{\sqrt{2}}\left(\chi_+ e^{-i\frac{E_+}{\hbar}t} + \chi_- e^{-i\frac{E_-}{\hbar}t} \right)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{E_+}{\hbar}t} \\ e^{-i\frac{E_-}{\hbar}t} \end{pmatrix}$$

* What is $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$?

$$\langle \chi | S_x | \chi \rangle = \frac{1}{2}\left(e^{i\frac{E_+}{\hbar}t}, e^{i\frac{E_-}{\hbar}t} \right) \times \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{E_+}{\hbar}t} \\ e^{-i\frac{E_-}{\hbar}t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\frac{E_+}{\hbar}t}, e^{i\frac{E_-}{\hbar}t} \right)\begin{pmatrix} e^{-i\frac{E_-}{\hbar}t} \\ e^{-i\frac{E_+}{\hbar}t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\frac{E_+ - E_-}{\hbar}t} + e^{-i\frac{E_+ - E_-}{\hbar}t} \right)$$

$$= \frac{\hbar}{2}\cos\left(\frac{E_+ - E_-}{\hbar}t \right)$$

$\frac{E_\pm}{\hbar} = \omega_\pm$

$$= \frac{\hbar}{2}\cos(\gamma B_0 t)$$

$$\langle \chi | S_y | \chi \rangle = i\frac{\hbar}{4}\left(e^{i\omega_+ t}, e^{i\omega_- t} \right) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_+ t} \\ e^{-i\omega_- t} \end{pmatrix}$$

$$= i\frac{\hbar}{4}\left(e^{i\omega_+ t},\ e^{i\omega_- t}\right)\begin{pmatrix} -e^{-i\omega_- t} \\ e^{-i\omega_+ t}\end{pmatrix}$$

$$= i\frac{\hbar}{4}\left(-e^{i(\omega_+ - \omega_-)t} + e^{-i(\omega_+ - \omega_-)t}\right)$$

$$= \frac{\hbar}{2}\sin\left((\omega_+ - \omega_-)t\right)$$

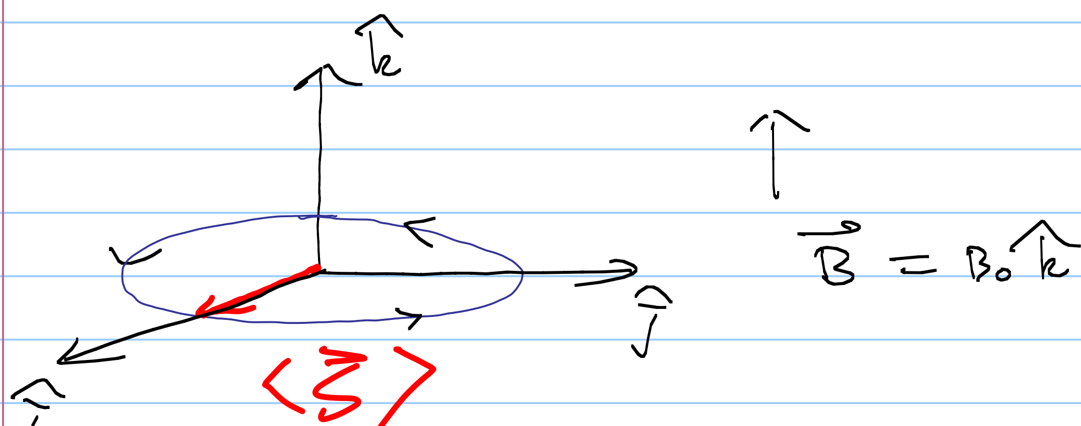
$$= \frac{\hbar}{2}\sin(\gamma B_0 t)$$

$$\langle \chi | S_z | \chi \rangle = \frac{\hbar}{4}\left(e^{i\omega_+ t},\ e^{i\omega_- t}\right) \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\omega_+ t} \\ e^{-i\omega_- t}\end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{0} - e^{0}\right) = 0$$

$$\langle \vec{S} \rangle = \langle S_x \rangle \hat{i} + \langle S_y \rangle \hat{j} + \langle S_z \rangle \hat{k}$$

$$= \frac{\hbar}{2}\left(\cos(\gamma B_0 t)\,\hat{i} + \sin(\gamma B_0 t)\,\hat{j}\right)$$



So the expectation value of $\vec{S}$ precesses around z axis as a function of time.

Addition of Angular Momentum

In many occasions, we need to consider the sum of two spin (or orbital) angular momentum;

$$\vec{S} \equiv \vec{S}^{(1)} + \vec{S}^{(2)}$$

In general, the possible total "S" values are

$$S = S_1 + S_2,\ S_1 + S_2 - 1,\ \cdots,\ |S_1 - S_2|$$

For the "m" values, because

$$S_z = S^{(1)}_z + S^{(2)}_z$$

$$\Rightarrow \quad m = m_1 + m_2$$

$\boxed{\text{Ex,}}$ Two spin $\frac{1}{2}$ particles form a composite particle. What are possible total spins?

$$S = \underbrace{\tfrac{1}{2} + \tfrac{1}{2}}_{\substack{\shortparallel \\ 1}},\ \underbrace{\tfrac{1}{2} - \tfrac{1}{2}}_{\substack{\shortparallel \\ 0}}$$

Now if $S^{(1)}_z \Rightarrow -\frac{\hbar}{2}$ $(m_1 = -\frac{1}{2})$ and $S_z^{(2)} \Rightarrow -\frac{\hbar}{2}$ $(m_2 = -\frac{1}{2})$

what will be the total S_z (m) ?
and total spin ?

$\Rightarrow m = -\frac{1}{2} - \frac{1}{2} = -1$

Since the possible s values are "1" and "0", but $s=0$ cannot yield $m=-1$, the only possible s value is "1"

In Dirac notation, this process can be written as follows.

spin $\frac{1}{2}$ particle 1 with $m_1 = -\frac{1}{2}$

$\Rightarrow |s_1\ m_1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$

spin $\frac{1}{2}$ particle 2 with $m_2 = -\frac{1}{2}$

$\Rightarrow |s_2\ m_2\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$

$\Rightarrow$ total spin of 1 with $m=-1$

$\Rightarrow |s\ m\rangle = |1, -1\rangle$

$\therefore |1, -1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle$

or equivalently $|1, -1\rangle = |\downarrow\rangle |\downarrow\rangle$

For more complicated cases, this process is tabulated by so-called Clebsch – Gordan coefficients.

We will not cover Clebsch – Gordan coefficients in this class, but it will be useful if you study how to use the Griffiths table 4.8.

Table 4.8 basically tells you how to add two different angular momentum quantities or decompose.

Spin III

Note Title 11/4/2009

In lecture Spin II, we have seen that

$$|s=1, m=-1\rangle = |s_1=\tfrac{1}{2}, m_1=-\tfrac{1}{2}\rangle |s_2=\tfrac{1}{2}, m_2=-\tfrac{1}{2}\rangle$$

In short

$$|1,-1\rangle = |\downarrow\rangle_1 |\downarrow\rangle_2 .$$

Then how are $|1,0\rangle$ and $|1,1\rangle$ expressed by the spin $\frac{1}{2}$ states ?

From $S_\pm = S^{(1)}_\pm + S^{(2)}_\pm$, and

$$S_+|1,-1\rangle = \hbar\sqrt{1(2)-(-1)0}\,|1,0\rangle$$
$$= \sqrt{2}\hbar\,|1,0\rangle$$

$$\Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}\hbar} S_+|1,-1\rangle = \frac{1}{\sqrt{2}\hbar}(S^{(1)}_+ + S^{(2)}_+)|1,-1\rangle$$

$$= \frac{1}{\sqrt{2}\hbar}(S^{(1)}_+ + S^{(2)}_+)|\downarrow\rangle_1|\downarrow\rangle_2$$

$$= \frac{1}{\sqrt{2}\hbar}\left(\hbar|\uparrow\rangle_1|\downarrow\rangle_2 + \hbar|\downarrow\rangle_1|\uparrow\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$$

, where I dropped "1" and "2", just to make it look simple.

If we apply $(S^{(1)}_+ + S^{(2)}_+)$ again, we will find $|1,1\rangle = |\uparrow\rangle|\uparrow\rangle$

These three states are called the triplet combination of two spin $\frac{1}{2}$ particles.

Another combination called singlet is expressed as

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \right)$$

Obviously, you can easily see that the right side should have $m_1 + m_2 = 0$.
Thus "m" total should be zero.
Now for "s" value,
if you apply $S_\pm = S^{(1)}_\pm + S^{(2)}_\pm$ to the right side, you will find that they are zero. This implies that the right side really has "s" value of zero as well.

This tells that the right side should be identical to $|0\,0\rangle$ up to a normalization constant.

* Combination of two spin $\frac{1}{2}$ particles appear very often, and thus should be understood well.

The bottom line is there are two states for each of the spin $\frac{1}{2}$ particles $\Rightarrow \{|\uparrow\rangle_1$ and $|\downarrow\rangle_1\}$ and $\{|\uparrow\rangle_2$ and $|\downarrow\rangle_2\}$, total of four states.

When they combine, the number of final states should still be four.

But they tend to group into two separate group of states.

One is the total spin "1" state called triple state

$$|1\,1\rangle = |\uparrow\rangle|\uparrow\rangle$$
$$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$$
$$|1\,{-1}\rangle = |\downarrow\rangle|\downarrow\rangle$$

And the other, total spin "0" state called singlet

$$|0\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

* It is important to note that when two spin $\frac{1}{2}$ particles are exchanged, the triplet states remain the same, but the singlet state changes its sign.